Wahl singularities in degenerations of del Pezzo surfaces

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Del Pezzo Wahl chains

A normal surface germ $(p \in X)$ is called a Wahl singularity if it is a quotient singularity and admits a $\mathbb{Q}$-Gorenstein smoothing with vanishing Milnor number. They are cyclic quotient singularities of the form: $\frac{1}{n^2}(1, na-1)$ with $0 < a < n$ and $gcd(a, n) = 1$.

These singularities appear predominantly in the study of 1-parameter $\mathbb{Q}$-Gorenstein smoothings of stable surfaces. By the work of Hacking and Prokhorov [2], all klt degenerations of $\mathbb{P}^2$ correspond to $\mathbb{Q}$-Gorenstein deformations of weighted projective planes $\mathbb{P}(a^2, b^2, c^2)$ where $a^2 + b^2 + c^2 = 3abc$. These surfaces contain only Wahl singularities. This classification can be recovered by means of Hirzebruch-Jung continued fractions.

For $0 < q < m$ with $gcd(m, q) = 1$, the fraction m/q is uniquely given by:

$$\frac{m}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_r}}} =: [e_1, \ldots, e_r].$$

Definition 1.1 A chain $[f_1, \ldots, f_r]$ with $f_i \geq 2$ admits a *zero continued fraction of weight* λ if there are indices $i_1 < i_2 < \ldots < i_v$ for some $v \geq 1$ and integers $d_{i_k} \geq 1$ such that

$$[\ldots, f_{i_1} - d_{i_1}, \ldots, f_{i_2} - d_{i_2}, \ldots, f_{i_v} - d_{i_v}, \ldots] = 0,$$

and $\lambda + 1 = \sum_{k=1}^{v} d_{i_k}$.

Proposition 1.2 ([6] Prop 4.1) A Wahl chain $\frac{n^2}{na-1} = [e_1, \ldots, e_r]$ appears in a degeneration of $\mathbb{P}^2$ if and only if there exists a $i \in \{1, \ldots, r\}$ such that $[e_1, \ldots, e_{i-1}]$ and $[e_{i+1}, \ldots, e_r]$ admit zero continued fractions of weight 0. (One $(i = 1, r)$ or both $(r = 1)$ chains could be empty.) In particular n is forced to be a Markov number.

We apply this principle to determine if a Wahl singularity appears in a $\mathbb{Q}$-Gorenstein degeneration of a del Pezzo surface.

Definition 1.3 A Wahl chain $[e_1, \ldots, e_r]$ is called *del Pezzo of type (I) or (II)* if it satisfies one of the following:

(I) $[e_1, \ldots, e_{r-1}]$ or $[e_2, \ldots, e_r]$ admits a zero continued fraction of weight $\lambda \leq 8$.

(II) There is $i \in \{2, \ldots, r-1\}$ such that $[e_1, \ldots, e_{i-1}]$ and $[e_{i+1}, \ldots, e_r]$ admit zero continued fractions of weights λ_1 and λ_2 such that $\lambda_1 + \lambda_2 \leq 8$.

Its *degree* is either $\ell = 9 - \lambda$ for (I), or $\ell = 9 - \lambda_1 - \lambda_2$ for (II). Its *marking* is the data of the zero continued fraction(s) involved in its type. If $[k_1, \ldots, k_{i-1}]$ and $[k_{i+1}, \ldots, k_r]$ are the corresponding zero continued fractions, then its marking is

$$[k_1, \ldots, k_{i-1}, \underline{e_i}, k_{i+1}, \ldots, k_r].$$

We call $\underline{e_i}$ the central mark.

Marked Del Pezzo surfaces

Given a del Pezzo Wahl chain $[e_1, \ldots, e_r]$ of degree ℓ and marking

$$[k_1, \ldots, k_{i-1}, \underline{e_i}, k_{i+1}, \ldots, k_r],$$

we construct its associated surface W_{*m} as follows:

- Consider the Hirzebruch surface $\mathbb{F}_{e_i}$.
- For type (II), choose two fibers of the $\mathbb{P}^1$-fibration $\mathbb{F}_{e_i} \to \mathbb{P}^1$ and perform blow-ups along these fibers to construct the chains for $[k_1, \ldots, k_{i-1}]$ and $[k_{i+1}, \ldots, k_r]$. For type (I) consider one fiber and perform the analogue blow-ups.
- Perform extra $\lambda_1 + 1$ and $\lambda_2 + 1$ (or just $\lambda + 1$) blow-ups at general points of the corresponding marked components. Then the Wahl chain is constructed.
- By contracting the Wahl chain, we obtain W_{*m}.

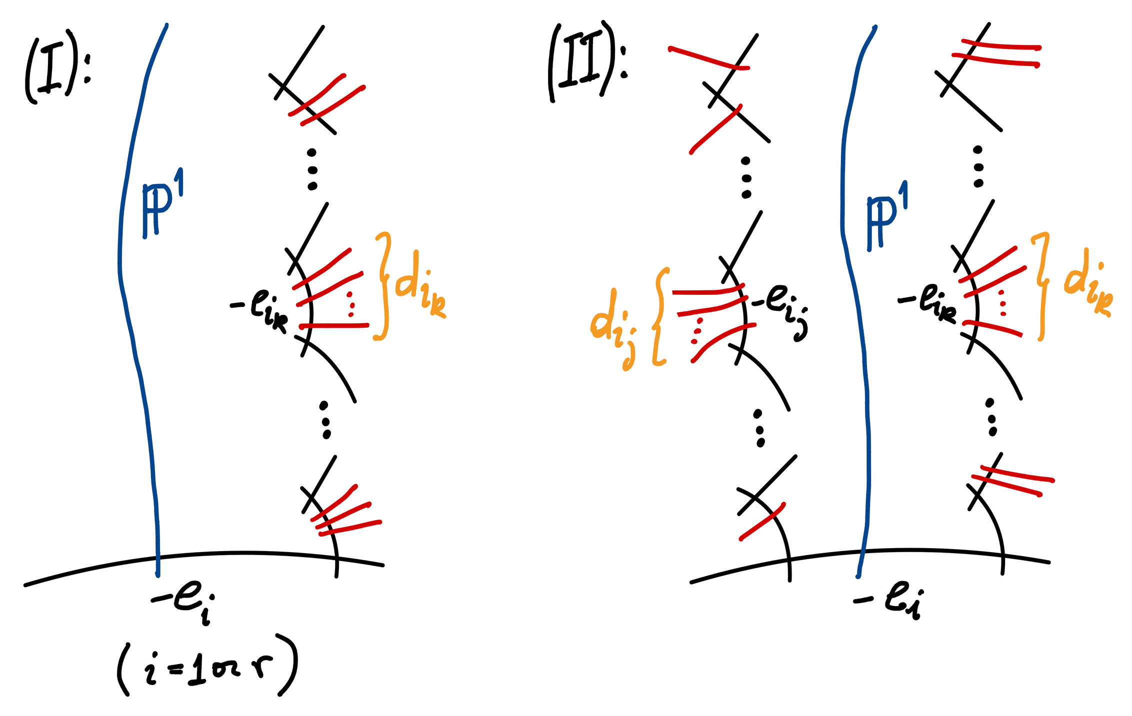

Figure 1. Minimal resolutions of W_{*m} for types (I) and (II).

Theorem 2.1: Let W_{*m} be a marked surface that is not of type (I) and degree 1. Then $-K_{W_{*m}}$ is an ample divisor. Moreover, there are no local-to-global obstructions to deform W_{*m}. Thus, there exists a $\mathbb{Q}$-Gorenstein smoothing $W_t \rightsquigarrow W_{*m}$ where W_t is a del Pezzo surface of degree ℓ.

Classification of W_* with $-K_{W_*}$ big and $K^2_{W^*} \geq 1$

Let W_* be a normal projective surface with a unique Wahl singularity, $-K_{W_*}$ and $K^2_{W^*} \geq 1$. Let $\phi : X \to W_*$ denote its minimal resolution and let $\pi : X \to \mathbb{F}_{d_{max}}$ a composition of blow-downs. We control the structure of $\pi(Exc(\phi))$ and show that W_* can be deformed into some $\widetilde{W_*}$ such that the corresponding configuration $\pi(Exc(\phi))$ consists of σ_∞ and at most two fibers f_0, f_1.

Furthermore, let $\widetilde{X} \to \widetilde{W_*}$ denote the minimal resolution. Then, there exists a family $(\widetilde{X} \subset \widetilde{\mathcal{X}}) \to (0 \in \mathbb{D})$, for which we perform a $K_{\widetilde{X}}$-MMP. It ends on a smooth deformation that further blowdowns to the deformation presented in the subsequent theorem.

Theorem 3.1 Let W_* be a normal projective surface with one Wahl singularity, $K^2_{W_*} \geq 1$, and $-K_{W_*}$ big. Then, there exists a $\mathbb{Q}$-Gorenstein deformation $Bl_{p_1, \cdots, p_u}(W_{*m}) \rightsquigarrow W_*$ for some smooth points $p_1, \cdots, p_u$, except for a single exceptional case.

This gives rise to the following consequences:

Corollary 3.2: There exists a $\mathbb{Q}$-Gorenstein smoothing $W_t \rightsquigarrow W$ such that $-K_{W_*}$ is ample, $K^2_{W_*} = \ell$, and $\frac{1}{n^2}(1, na-1) \in W_*$ if and only if the Wahl chain of $\frac{n^2}{na-1}$ is a del Pezzo chain of degree $\geq \ell$. Moreover,

- Every Wahl singularity appears in a $\mathbb{Q}$-Gorenstein smoothing $W_t \rightsquigarrow W_*$, where W_t is a del Pezzo surface of degree $\ell \leq 4$.
- There exist infinitely many Wahl singularities that do not appear in a $\mathbb{Q}$-Gorenstein smoothing $W_t \rightsquigarrow W$, where W_t is a del Pezzo surface of degree $\ell \geq 5$.

Additionally, the surfaces W_{*m} fit in the picture of the toric degenerations of del Pezzo surfaces. Any pair of toric del Pezzo surfaces with T-singularities and fixed degree ℓ are connected by a tree of smooth rational curves [3, Theorem 1]. Through a special type of deformation which we refer as *slides* (see [7, Section 4]), we prove the following statement:

Theorem 3.3: There exists a canonically defined del Pezzo toric surface W^T_{*m}, such that W^T_{*m} contains at most T-singularities and W_{*m} admits a $\mathbb{Q}$-Gorenstein degeneration to W^T_{*m}.

Applications to vector bundles on del Pezzo surfaces

Definition 4.1: An exceptional vector bundle E (e.v.b) on a projective surface Y is a locally free sheaf such that $Hom(E, E) = \mathbb{C}$, and $Ext^1(E, E) = Ext^2(E, E) = 0$. An *exceptional collection of vector bundles* (e.c) on a projective surface Y is a collection of exceptional vector bundles $E_r, E_{r-1}, \ldots, E_0$ such that $Ext^k(E_i, E_j) = 0$ for all $i < j$ and all $k \geq 0$.

Hacking in [1, Theorem 1.1], provides a technique to construct exceptional vector bundles on $\mathbb{Q}$-Gorenstein smoothings of surfaces with a unique Wahl singularity and $p_g = q = 0$. This was further generalized for multiple singularities by Tevelev-Urzúa in [5].

Theorem 4.2 ([5] Thm 5.5 and 5.8) Let $W_t \rightsquigarrow W$ be a $\mathbb{Q}$-Gorenstein smoothing of a normal projective surface with at most Wahl singularities such that:

(1) $p_g(W) = q(W) = 0$,

(2) The surface W has exactly the Wahl singularities $p_0, \ldots, p_r$ (we also allow P_i to be smooth points), and a chain of nonsingular rational curves $\Gamma_1, \ldots, \Gamma_r$ that are toric boundary divisors Γ_i, Γ_{i+1} at p_{i+1}, and

(3) There exists a Weil divisor $A \subset W$, which is Cartier outside of p_0 and generates the local class group $(p_0 \in W)$.

Then, after possibly shrinking $\mathbb{D}$, there exists an (e.c) $E_r, \ldots, E_0$ on W_t with $t \neq 0$, such that $rk(E_i) = n_i$ and $c_1(E_i) = -n_i(A + \Gamma_1 + \ldots + \Gamma_i) \in H^2(W_t, \mathbb{Z})$, where $p_i = \frac{1}{n_i^2}(1, n_i a_i - 1)$.

By combining the previous theorem and Corollary 3.2, we reprove the following result of Polishchuck-Rains [4, Theorem A].

Theorem 4.2 Let ∂, n be coprime integers with $n > 0$. Then there is an e.c. $E, \mathcal{O}_Y$ on a del Pezzo surface Y of degree 4 with $rk(E) = n$ and $deg(E) = \partial$.

Furthermore, we obtain:

Theorem 4.3: There exist infinitely many pairs ∂, n of coprime integers with $0 < \partial < n$ such that an e.v.b. E with $rk(E) = n$ and $deg(E) \equiv \pm\partial \pmod{n}$ is not realizable on any del Pezzo surface of degree ≥ 5.

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